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A GENERAL THEORY OF FIRST-PASSAGE DISTRIBUTIONS IN TRANSPORT AND MULTIPLICATIVE PROCESSES

by

J. E. Moyal

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J. E. Moyal

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ABSTRACT

The "Milne problem," expressed in probabilistic terms, is solved for general transport and multiplicative processes. If a particle initially in a given state at a given position inside a surface τ is multiply scattered while traveling through a fixed medium, then, given the scattering cross sections and, if required, the probability distribution for a change of state between collisions (e.g., by diffusion or ionization), the problem is to obtain the probability that the particle eventually effects a first passage through a specified position on the surface τ and in a specified state. In the case of a multiplicative process, given, in addition, the rates of creation and annihilation of particles (considering the nature of the particle as a state variable), the problem is to obtain the probability that eventually n particles will emerge for the first time through specified positions on τ and in specified states (with $n = 0, 1, 2, \ldots$). A general solution is given in the form of a convergent series whose terms are obtained by iteration; this solution is unique if and only if the probability θ_∞ of an infinity of atomic events before a first passage (which is the limit of a certain nonincreasing sequence) is identically zero; in the multiplicative case, $\theta_\infty \not\equiv 0$ may be taken to mean that the process is "supercritical." The mathematical theory that leads to this solution is a generalization of the corresponding theory for time-dependent Markov processes in which the time variable is replaced by a set of surfaces ordered by inclusion of their "insides," and is valid for Euclidean space of any number of dimensions. Applying it to the four-dimensional space of special relativity with ordered sets of space-like surfaces, one obtains a Lorentz-invariant formulation of the theory of physical Markov processes. A few examples are given.

ABSTRACT

The "Milne problem," expressed in probabilistic terms, is solved for general transport and multiplicative processes. If a particle initially in a given state at a given position inside a surface T is multiply scattered while traveling through a fixed medium, then, given the scattering cross sections and, if required, the probability distribution for a change of state between collisions (e.g., by diffusion or ionization), the problem is to obtain the probability that the particle eventually effects a first passage through a specified position on the surface T and in a specified state. In the case of a multiplicative process, given, in addition, the rates of creation and annihilation of particles (considering the nature of the particle as a state variable), the problem is to obtain the probability that eventually n particles will emerge for the first time through specified positions on T and in specified states (with $n = 0, 1, 2, \ldots$). A general solution is given in the form of a convergent series whose terms are obtained by iteration; this solution is unique if and only if the probability θ_∞ of an infinity of atomic events before a first passage (which is the limit of a certain nonincreasing sequence) is identically zero; in the multiplicative case, $\theta_\infty \neq 0$ may be taken to mean that the process is "supercritical." The mathematical theory that leads to this solution is a generalization of the corresponding theory for time-dependent Markov processes in which the time variable is replaced by a set of surfaces ordered by inclusion of their "insides," and is valid for Euclidean space of any number of dimensions. Applying it to the four-dimensional space of special relativity with ordered sets of space-like surfaces, one obtains a Lorentz-invariant formulation of the theory of physical Markov processes. A few examples are given.

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I. INTRODUCTION

Many physical processes have the Markovian character in which a particle suffers a succession of independent random scatterings while traveling through some medium. Examples are: the diffuse scattering of light, where the particle is a photon, the diffusion of neutrons, and the multiple scattering of charged particles. In connection with such processes, one is often interested in the probability distribution of the state variables of the particle (velocity, energy, spin, etc.) and of the position at which it emerges on its first passage through some surface independently of time. The position probability will yield, by integration, the average flux density in the case of a source or beam of particles. The theory of such first-passage distributions goes by the name of theory of radiative transfer in the case of the scattering of light (c.f. Chandrasekhar[3] and Sobolev;[9] see also Wing,[10] where such problems are discussed in a more general context). The problem of obtaining the first-passage distributions, given the microscopic scattering laws, is essentially the well-known Milne problem, which has been solved exactly under rather restrictive simplifying assumptions by means of the Wiener-Hopf integral-equation technique [c.f. Busbridge[2]]. The purpose of this report is to present a general theory for the solution of such first-passage problems. We shall show that a solution always exists and present it in the form of a convergent series whose terms are obtained by iteration, and we shall give necessary and sufficient conditions for this solution to be unique. Furthermore, we shall see that the theory developed for this purpose can also be made to yield a Lorentz-invariant formulation of the basic equations of physical Markov processes. Finally, we shall show that the theory generalizes to first-passage problems in the case of processes involving the creation and annihilation of particles, such as the multiplication of neutrons in fissionable material and electron-photon or nucleon cascades.

II. FIRST-PASSAGE PROCESSES

The theory we are going to develop is based on a generalization of the concept of a Markov process: such a process is usually defined on a linearly ordered set (the time axis), and the required generalization

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II. FIRST-PASSAGE PROCESSES

The theory we are going to develop is based on a generalization of the concept of a Markov process: such a process is usually defined on a linearly ordered set (the time axis), and the required generalization consists in extending this definition to a partially ordered set. Let $\mathscr{T}$ be such a set, and let $\tau \geq \tau_0$ denote the ordering relation. For the purposes of the present theory, $\mathscr{T}$ will consist of a set of two-sided, oriented, continuous and simply-connected surfaces in some finite-dimensional Euclidean space X. Each such surface τ partitions X into two disjoint sets: the "inside" X_τ of τ (which conventionally will include τ), and the "outside" X'_τ of τ. We partially order $\mathscr{T}$ by inclusion of the "insides" of its elements; i.e., we set $\tau \geq \tau_0$ whenever $X_\tau \supseteq X_{\tau_0}$. In the applications to scattering processes, $X = R_3$; in the Lorentz-invariant formulation of Markov process theory, $X = R_4$. The whole theory can be extended to more general topological spaces than R_n, but we shall not consider this generalization here. To each $\tau \in \mathscr{T}$ is assigned a space Ω_τ of elementary events ω_τ and a σ-field $\mathscr{F}_\tau$ of measurable subsets Γ_τ of Ω_τ. If $\mathscr{A}$ is the set of all possible states α of the particle (velocity, energy, spin, etc.), and if x_τ denotes the position at which it emerges on its first passage through the surface τ, then $\omega_\tau = (\alpha, x_\tau)$ and $\Omega_\tau = \mathscr{A} \times \tau$, which is a subset of the space $\Omega = \mathscr{A} \times X$. We assume given σ-fields $\mathscr{B}_\mathscr{A}$ of sets A in $\mathscr{A}$ and $\mathscr{B}_X$ of sets S in X, such that $\mathscr{T} \subseteq \mathscr{B}_X$ (i.e., each surface τ is measurable). It follows that the class of all measurable subsets S_τ of τ is itself a σ-field $\mathscr{B}_\tau$, and we set $\mathscr{F}_\tau = \mathscr{B}_\mathscr{A} \times \mathscr{B}_\tau$, which is a subfield of the σ-field $\mathscr{F} = \mathscr{B}_\mathscr{A} \times \mathscr{B}_X$. Suppose furthermore that, for each ordered pair $\tau \geq \tau_0$ of elements of $\mathscr{T}$, we have defined a function P on $\mathscr{F}_\tau \times \Omega_{\tau_0}$ so that for each fixed $\omega_{\tau_0} \in \Omega_{\tau_0}$, $P(\cdot \mid \omega_{\tau_0})$ is a measure on $\mathscr{F}_\tau$ satisfying the normalization condition,

$$\kappa(\tau|\omega_{\tau_0}) = P(\Omega_\tau|\omega_{\tau_0}) \leq 1, \tag{2.1}$$

and for each fixed set $\Gamma_\tau \in \mathscr{F}_\tau$, $P(\Gamma_\tau|\cdot)$ is a measurable function on Ω_{τ_0}. Then P has the character of an incomplete conditional-probability distribution (incomplete in the sense that it is not normalized to unity). For our present purposes, P is interpreted as follows: given that the particle is initially at the point $x_{\tau_0} \in X_\tau$ in the state α_0, $P(A \times S_\tau|\alpha_0, x_{\tau_0})$ is the probability that it effects a first passage through some point of the set $S_\tau \subseteq \tau$ in some state $\alpha \in A$. Hence, $\kappa(\tau|\alpha_0, x_{\tau_0})$ is the total probability of a first passage through τ and $\eta = 1 - \kappa$ is the probability that the particle never passes through τ; η will in general be the sum of a stopping probability σ (due to slowing down of the particle in the scattering medium) and an escape probability ϵ (due to the particle escaping to infinity if τ is not closed). For this reason, we shall call P a first-passage distribution and η a no-passage probability; it is precisely because we have to allow for processes with nonzero η that we do not require P to be normalized to unity.

We now say that the family of all first-passage distributions defines a generalized Markov process over the partially-ordered set $\mathscr{T}$ if its elements P satisfy the Chapman-Kolmogorov relation [illegible] in extending this definition to a partially ordered set. Let $\mathcal{T}$ be such a set and let [illegible] denote the ordering relation. For the purposes of the present theory, $\mathcal{T}$ will consist of a set of two-sided, oriented, continuous and simply-connected surfaces lying in [illegible]-dimensional Euclidean space X, each such surface [illegible] partition X [illegible] disjoint sets [illegible] the "inside" [illegible] and [illegible] which conventionally will [illegible] the "outside" X_t^{+} of t. We partially order $\mathcal{T}$ [illegible] of its elements, i.e., [illegible] whenever [illegible] [illegible] to arbitrary processes [illegible] $X = R_1$ in the [illegible] Markov process [illegible] $X = R_2$. The [illegible] more general topological spaces than [illegible] generalization here [illegible] each [illegible] $\mathcal{T}$ is assumed [illegible] space [illegible] elementary events ω and a [illegible] field $\mathcal{F}_{\omega}$ of measurable [illegible] set of all possible [illegible] of the particle [illegible] and u [illegible] denotes the position at which [illegible] first passage through the surface t. Then [illegible] $= (u, x)$ and [illegible] which is a subset of the space $\Omega = \mathcal{A} \times X$. We assume given [illegible] $\mathcal{B}_{\mathcal{A}}$ of sets A in $\mathcal{A}$ and $\mathcal{B}_X$ of sets S in X_t such that $\mathcal{F} \subseteq \mathcal{B}_X$ [illegible] each surface [illegible] is measurable. It follows that the class of all measurable subsets [illegible] of [illegible] is a [illegible] field $\mathcal{F}_t$ and we set [illegible] $= \mathcal{B}_{\mathcal{A}} \times S_t$, which is a subfield of the field $\mathcal{F} = \mathcal{B}_{\mathcal{A}} \times \mathcal{B}_X$. Suppose furthermore that [illegible] each ordered pair [illegible] of elements of $\mathcal{T}$, we have defined a function P on $\mathcal{F} \times [illegible]$, so that for each fixed [illegible], $P(\cdot \mid \cdot, \cdot)$ is a measure on $\mathcal{F}$ satisfying the normalization condition

$$
x(\cdot \mid \cdot) \ldots \left(\Omega_{t_2}\right) \ldots \qquad (2.1)
$$

and for each fixed [illegible] $\in \mathcal{G}$, $P(\cdot \mid \cdot)$ is a measurable function on [illegible]. Then P has the character of an incomplete conditional-probability distribution (incomplete in the sense that it [illegible] not necessarily [illegible] unity). For our present purposes P is interpreted as follows: given that the particle is initially at the point [illegible] X in the state [illegible], $P(\cdot \mid \cdot)$ is the probability that it effects a first passage through [illegible] some point of the set S [illegible] in some state $u \in A$ [illegible] ($P(\cdot, x)$ is the total probability of a first passage through [illegible] [illegible] $1 - \ldots$ is the probability that the particle never passes through [illegible] which may in general be the result of a stopping probability q (due to slowing down of the particle in the scattering medium) [illegible] or the probability [illegible] that the particle escapes [illegible] not closed. For this reason, we shall call P a first-[illegible] distribution and [illegible] non-passage probability. It is precisely because [illegible] processes with nonzero [illegible] that we do not require $\mathcal{F}$ to be nonnegatively [illegible].

We now say that the family of all first-passage distributions defines a generalized Markov process over the partially ordered set $\mathcal{T}$ if it satisfies P satisfy the [illegible] Kolmogorov relation

$$P(\Gamma_\tau | \omega_{\tau_0}) = \int_{\Omega_{\tau_1}} P(\Gamma_\tau | \omega_{\tau_1}) P(d\omega_{\tau_1} | \omega_{\tau_0}) \tag{2.2}$$

for every ordered triple $\tau \geq \tau_1 \geq \tau_0$. It follows from this definition that the transition distribution P defines an ordinary incomplete Markov process over every linearly-ordered chain in $\mathcal{T}$. We may therefore regard such a generalized process as a family of ordinary Markov processes over the chains in $\mathcal{T}$ mutually related by (2.2). It is clear that this family does not define a stochastic process over $\mathcal{T}$ in the usual sense, because the conditional distributions P are not defined for pairs of elements of $\mathcal{T}$ which are not related by ordering. For the purposes of this report, where P has the interpretation outlined above, we shall call such a generalized Markov process a first-passage process.

III. LORENTZ-INVARIANT FORMULATION OF MARKOV PROCESS EQUATIONS

Suppose now that X is the Minkowski four-dimensional space-time of special relativity R_4, and take $\mathcal{B}_X$ to be the Borel field of subsets of R_4. If we choose $\mathcal{T}$ to be the set of all space-like surfaces in R_4 satisfying the assumptions made above, then it is clear that the definition above yields a relativistically invariant formulation of the concept of a temporal-particle Markov process, in the sense that the Chapman-Kolmogorov relation (2.2) is then invariant under Lorentz transformations. The same will be true, as we shall see later, of other basic equations, such as the integral equation (4.2) and the "backward" integro-differential equation (5.6). If we choose a particular Galileian frame of reference L in R_4, and consider the linear chain $\mathcal{T}_L$ in $\mathcal{T}$ consisting of all flat space-like surfaces τ normal to the time-axis in L, then clearly we can assimilate τ to the time coordinate and $\mathcal{T}_L$ to the time axis in L; x_τ is then a point x, S_τ is a Borel subset S of three-dimensional space R_3 at the time τ, and the first-passage distribution P, restricted to $\mathcal{T}_L$, defines an ordinary temporal Markov process over $\mathcal{T}_L$, with τ as parameter, satisfying the usual Chapman-Kolmogorov relation.

$$P(A \times S; \tau | \alpha_0, x_0; \tau_0) = \int_{\mathcal{A} \times R_3} P(A \times S; \tau | \alpha_1, x_1; \tau_1)\, P(d\alpha_1 dx_1; \tau_1 | \alpha_0, x_0; \tau_0),$$

$$(\tau \geq \tau_1 \geq \tau_0). \tag{3.1}$$

The no-passage probability η defined in Section II, if it is not identically zero, must clearly be interpreted in this context as the cumulative distribution of the lifetime of the particle; i.e., $\eta(\tau | \alpha_0, x_0 \tau_0)$ is the probability

$$P(\Gamma_{\tau}|\omega_{\tau_0}) = \int_{\Omega_{\tau_1}} P(\Gamma_{\tau}|\omega_{\tau_1})\, P(d\omega_{\tau_1}|\omega_{\tau_0}) \qquad (2.2)$$

for every ordered triple $\tau \geq \tau_1 \geq \tau_0$. It follows from this definition that the transition distribution P defines an ordinary incomplete Markov process over every linearly-ordered chain in $\mathcal{T}$. We may therefore regard such a generalized process as a family of ordinary Markov processes over the chains in $\mathcal{T}$ mutually related by (2.2). It is clear that this family does not define a stochastic process over $\mathcal{T}$ in the usual sense, because the conditional distributions P are not defined for pairs of elements of $\mathcal{T}$ which are not related by ordering. For the purposes of this report, where P has the interpretation outlined above, we shall call such a generalized Markov process a first-passage process.

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$$P(A \times S;\tau|a_0,x_0;\tau_0) = \int_{A\times R_3} P(A \times S,\tau|a_1,x_1;\tau_1)\, P(da_1 dx_1;\tau_1|a_0,x_0;\tau_0)$$

$$(\tau \geq \tau_1 \geq \tau_0). \qquad (3.1)$$

The no-passage probability η defined in Section II, if it is not identically zero, must clearly be interpreted in this context as the cumulative distribution of the lifetime of the particle; i.e., $\eta(\tau|a_0,x_0;\tau_0)$ is the probability that the particle initially in state α_0 and position x_0 at time τ_0 is annihilated at some time $t \leq \tau$. We remark that a Lorentz-invariant formulation of this kind is not appropriate in the case of multiple-scattering processes of the type studied in the sections that follow, because for such processes there exists an obviously preferred class of reference systems, namely, those in which the scattering medium is at rest.

IV. DISCONTINUOUS FIRST-PASSAGE PROCESSES

In this section, we extend to first-passage processes the theory of discontinuous Markov processes developed in Moyal[5,8] (referred to henceforth as I and II, respectively). The type of process we have in mind is one in which the particle suffers multiple collisions, each causing an instantaneous change in its state. We assume that between collisions the process is governed by a known first-passage distribution P_0. More precisely, $P_0(A \times S_\tau | \alpha_0, x_{\tau_0})$ is the probability that the particle, initially in a state α_0 at the point $x_{\tau_0} \in \mathcal{X}_\tau$ (the "interior" of τ), effects a first passage through S_τ in some state $\alpha \in A$ before it has suffered any collision. We assume that P_0 satisfies the Chapman-Kolmogorov relation (2.2), so that it defines a first-passage process dependent on no collisions. This formulation has the virtue of including processes in which not only the position, but also the state, of the particle can change between collisions. For example, P_0 may characterize changes of velocity by diffusion, or loss of energy by ionization. If, between collisions, the particle moves in a straight line with its velocity and other state variables remaining constant, then we have a purely discontinuous (or pure multiple scattering) process; this case is dealt with in greater detail in Section V. The effect of the collisions is assumed to be specified by a known first collision and consequent state distribution Q, which is a conditional distribution on $\mathcal{F} \times \Omega$. $Q(A \times X; \tau | \alpha_0, x_{\tau_0})$ is interpreted as the probability that the particle, initially in state α_0 at $x_{\tau_0} \in \mathcal{X}_\tau$, suffers its first collision at some point $x \in X$ (where X is a measurable subset of $\mathcal{X}$) before it has made a first passage through the surface τ, and that its state immediately after this first collision is some $\alpha \in A$. It follows from this definition that

$$Q\left(A \times X; \tau | \alpha_0, x_{\tau_0}\right) \equiv Q\left(A \times (X \cap \mathcal{X}_\tau); \tau | \alpha_0, x_{\tau_0}\right).$$

The first-passage distribution P_0 and the conditional distribution Q are assumed to be related as follows: for every ordered triple $\tau \geq \tau_1 \geq \tau_0$,

$$Q\left(A \times X; \tau | \alpha_0, x_{\tau_0}\right) = Q\left(A \times X; \tau_1 | \alpha_0, x_{\tau_0}\right) + \int_{\mathcal{A} \times \tau} Q\left(A \times X; \tau | \alpha_1, x_{\tau_1}\right) P_0\left(d\alpha_1 dx_{\tau_1} | \alpha_0, x_{\tau_0}\right). \tag{4.1}$$

that the particle initially in state a_0 and position x_0 at time τ_0 is annihilated at some time $t \leq \tau$. We remark that a Lorentz-invariant formulation of this kind is not appropriate in the case of multiple-scattering processes of the type studied in the sections that follow, because for such processes there exists an obviously preferred class of reference systems, namely, those in which the scattering medium is at rest.

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$$Q(A \times X; \tau | a_0, x_{\tau_0}) = Q(A \times (X \cap X_\tau); \tau | a_0, x_{\tau_0}).$$

The first-passage distribution P_0 and the conditional distribution Q are assumed to be related as follows: for every ordered triple $\tau \geq \tau_1 \geq \tau_0$,

$$Q(A \times X; \tau | a_0, x_{\tau_0}) = Q(A \times X; \tau_1 | a_0, x_{\tau_0})$$

$$+ \int_{A \times S_{\tau_1}} Q(A \times X; \tau | a_1, x_{\tau_1}) \, P_0(da_1 dx_{\tau_1} | a_0, x_{\tau_0}). \tag{4.1}$$

The intuitive meaning of this relation is that the first collision and consequent state probability in its left-hand side, which depends on the particle not making a first passage through τ, is the sum of two first collisions and consequent state probabilities. The first (which is the first term in its right-hand side) is dependent on no first passage through $\tau_1 \leq \tau$; the second (which is the second term in its right-hand side) is dependent on a first passage through τ_1 without collision and no first passage through τ.

The first-passage distribution P we are seeking must then satisfy the following integral equation:

$$P\left(A \times S_\tau | \alpha_0, x_{\tau_0}\right) = P_0\left(A \times S_\tau | \alpha_0, x_{\tau_0}\right) + \int_{\mathcal{A} \times X_\tau} P(A \times S_\tau | \alpha, x)\, Q(d\alpha dx | \alpha_0, x_{\tau_0}), \tag{4.2}$$

which we also write in the abbreviated notation $P = P_0 + P * Q$, where the symbol $*$ stands for the integration operation that occurs in the second term in the right-hand side of (4.2). The intuitive meaning of this equation is, that the first-passage distribution P in its left-hand side is the sum of two first-passage distributions: the first P_0 with no collisions, and the second (which is the second term in its right-hand side) with at least one collision. The problem that now confronts us is that of the existence of a solution of (4.2) which is a first-passage distribution satisfying the Chapman-Kolmogorov relation (2.2), and of the conditions under which this solution is unique.

We define as in I and II, two sequences $\{Q_n\}$, $\{P_n\}$, where $Q_1 = Q$, $Q_{n+1} = Q_n * Q$, and $P_n = P_0 * Q_n$, $n = 1, 2, \ldots$. If the particle is initially in the state α_0 at $x_{\tau_0} \in X_\tau$, then $Q_n(A \times X; \tau | \alpha_0, x_{\tau_0})$ represents the probability that the nth collision occurs in X with consequent state $\alpha \in A$ before the particle effects a first passage through τ. Therefore

$$\theta_n(\tau | \alpha_0, x_{\tau_0}) = Q_n(\mathcal{A} \times X; \tau | \alpha_0, x_{\tau_0})$$

represents the probability that the particle suffers at least n collisions in X_τ before it effects a first passage through τ. $P_n(A \times S_\tau | \alpha_0, x_{\tau_0})$ represents the probability that the particle effects a first passage through S_τ while in a state $\alpha \in A$ and after suffering exactly n collisions in X_τ. Therefore

$$\kappa_n\left(\tau | \alpha_0, x_{\tau_0}\right) = P_n\left(\mathcal{A} \times \tau | \alpha_0, x_{\tau_0}\right)$$

represents the probability of a first passage through τ after exactly n collisions in X_τ. Let $\eta_0 = 1 - \kappa_0 - \theta_1$, and let $\eta_n = \eta_0 * Q_n$, $n = 1, 2, \ldots$. It is then easy to see that $\eta_n(\tau | \alpha_0, x_{\tau_0})$ is the probability that the particle suffers

The intuitive meaning of this relation is that the first collision and consequent state probability in its left-hand side, which depends on the particle not making a first passage through τ, is the sum of two first collisions and consequent state probabilities. The first (which is the first term in its right-hand side) is dependent on no first passage through $\tau_1 \le \tau$; the second (which is the second term in its right-hand side) is dependent on a first passage through τ_1 without collision and no first passage through τ.

The first-passage distribution P we are seeking must then satisfy the following integral equation:

$$P\left(A \times S_\tau \middle| \alpha_0, x_{\tau_0}\right) = P_0\left(A \times S_\tau \middle| \alpha_0, x_{\tau_0}\right)$$

$$+ \int_{\mathcal{A} \times X_{\tau_0}} P(A \times S_\tau | \alpha, x)\, Q(d\alpha \times dx | \alpha_0, x_{\tau_0}), \qquad (4.2)$$

which we also write in the abbreviated notation $P = P_0 + P * Q$, where the symbol $*$ stands for the integration operation that occurs in the second term in the right-hand side of (4.2). The intuitive meaning of this equation is, that the first-passage distribution P in its left-hand side is the sum of two first-passage distributions: the first P_0 with no collisions, and the second (which is the second term in its right-hand side) with at least one collision. The problem that now confronts us is that of the existence of a solution of (4.2) which is a first-passage distribution satisfying the Chapman-Kolmogorov relation (2.2), and of the conditions under which this solution is unique.

We define as in I and II two sequences (Q_n), (P_n), where $Q_1 = Q$, $Q_{n+1} = Q_n * Q$, and $P_n = P_0 * Q_n$, $n = 1, 2, \ldots$. If the particle is initially in the state α_0 at $x_{\tau_0} \in X_{\tau_0}$, then $Q_n(A \times X_\tau | \alpha_0, x_{\tau_0})$ represents the probability that the nth collision occurs in X_τ with consequent state $\alpha \in A$ before the particle effects a first passage through τ. Therefore

$$\theta_n(\tau | \alpha_0, x_{\tau_0}) = Q_n(\mathcal{A} \times X_\tau | \alpha_0, x_{\tau_0})$$

represents the probability that the particle suffers at least n collisions in X_τ before it effects a first passage through τ. $P_n(A \times S_\tau | \alpha_0, x_{\tau_0})$ represents the probability that the particle effects a first passage through S_τ while in a state $\alpha \in A$ and after suffering exactly n collisions in X_τ. Therefore

$$\pi_n\left(\tau | \alpha_0, x_{\tau_0}\right) = P_n\left(\mathcal{A} \times S_\tau | \alpha_0, x_{\tau_0}\right)$$

represents the probability of a first passage through τ after exactly n collisions in X_τ. Let $\eta_0 = 1 - \theta_0 - \pi_0$, and let $\eta_n = \eta_0 * Q_n$, $n = 1, 2, \ldots$. It is then easy to see that $\eta_n(\tau | \alpha_0, x_{\tau_0})$ is the probability that the particle suffers exactly n collisions without ever making a first passage through τ. One can then show almost exactly, as in the proof of (2.4) in II, that

$$\theta_n = 1 - \sum_{i=0}^{n-1} (\kappa_i + \eta_i) \le 1, \tag{4.3}$$

and that the sequence $\{\theta_n\}$ is nondecreasing. Let

$$\theta_\infty = \lim_{n\to\infty} \theta_n = 1 - \sum_{i=0}^{\infty} (\kappa_i + \eta_i) \le 1. \tag{4.4}$$

We see from (4.4) that $\Sigma\kappa_n$ and $\Sigma\eta_n$ both converge. The series ΣP_n is obviously majorized by $\Sigma\kappa_n$ and is hence convergent. Let

$$P_R = \sum_{n=0}^{\infty} P_n. \tag{4.5}$$

We then prove almost precisely as in the proof of Theorem 6.1 of I that P_R is a first-passage distribution satisfying the Chapman-Kolmogorov relation (2.2) and the integral equation (4.2); we call P_R the regular solution of (4.2).

One can show, as in the proof of Theorem 8.3 of I, that the regular solution P_R is the minimal nonnegative solution of (4.2). Whether it is also its unique solution hinges on the values of θ_∞. It is unique if and only if θ_∞ is identically zero; this is shown as in the proof of the corollary to Theorem 6.2 of I. We call the process stable when $\theta_\infty \equiv 0$. These results are summarized in the following theorem:

Theorem 4.1. The series

$$\sum_{n=0}^{\infty} P_n$$

converges to a first-passage distribution P_R which satisfies the Chapman-Kolmogorov relation (2.2) and is the minimal nonnegative solution of the integral equation (4.2); moreover, P_R is the unique solution of (4.2) if and only if $\theta_\infty \equiv 0$.

Let $\kappa_R = \Sigma_0^\infty \kappa_n$ and $\eta_R = \Sigma_0^\infty \eta_n$; then clearly

$$\kappa_R(\tau|\alpha_0, x_{\tau_0}) = P_R(\mathcal{A} \times \tau|\alpha_0, x_{\tau_0}),$$

exactly n collisions without ever making a first passage through r. One can then show almost exactly, as in the proof of (2.4) in II, that

$$\theta_n = 1 + \sum_{i=0}^{n-1} (\kappa_i + \eta_i) \leq 1, \tag{4.3}$$

and that the sequence $\{\theta_n\}$ is nondecreasing. Let

$$\theta_\infty = \lim_{n\to\infty} \theta_n = 1 + \sum_{i=0}^{\infty} (\kappa_i + \eta_i) \leq 1. \tag{4.4}$$

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Theorem 4.1. The series

$$\sum_{n=0}^{\infty} P_n$$

converges to a first-passage distribution P_R which satisfies the Chapman-Kolmogorov relation (2.2) and is the minimal nonnegative solution of the integral equation (4.2); moreover, P_R is the unique solution of (4.2) if and only if $\theta_\infty = 0$.

Let $\kappa_R = \Sigma_0^\infty \kappa_n$ and $\eta_R = \Sigma_0^\infty \eta_n$; then clearly

$$\kappa_R(r|x_0, r_0) = P_R(\mathcal{M} \times r|x_0, r_0),$$

and it follows from (4.4) that the corresponding no-passage probability

$$\eta = 1 - \kappa_R = \eta_R + \theta_\infty.$$

Thus, if the particle is initially in the state α_0 at x_{τ_0}, then the no-passage probability $\eta(\tau|\alpha_0, x_{\tau_0})$ is the sum of two probabilities, $\eta_R(\tau|\alpha_0, x_{\tau_0})$ and $\theta_\infty(\tau|\alpha_0, x_{\tau_0})$. Clearly, η_R represents the stopping or escape probability after a finite number of collisions in $\mathbf{X}_\tau$. Hence, θ_∞ must be interpreted as the stopping or escape probability after an infinite number of collisions. Note that the process is stable if and only if $\eta = \eta_R$. It is also clear that $\eta_R \equiv 0$ if and only if the no-passage probability with no collisions $\eta_0 \equiv 0$. Hence, the total no-passage probability $\eta \equiv 0$ if and only if both $\eta_0 \equiv 0$ and $\theta_\infty \equiv 0$.

V. FIRST-PASSAGE DISTRIBUTIONS IN PURE MULTIPLE SCATTERING PROCESSES

In this section, we apply the theory outlined in Section IV to the special case of a pure multiple-scattering process in R_3, where only the position of the particle changes between collisions, the other state variables remaining constant. Let μ represent the unit vector in the direction of motion of the particle. If the particle initially at x suffers no collisions while traveling a length of path s, then its position vector becomes $x + \mu s$. It is convenient to distinguish μ from the remaining state variables γ; thus, $\alpha = (\gamma, \mu)$, and $\mathcal{A} = G \times M$, where G is the set of all γ, and M is the set of all directions μ. We assume that the probability that a particle at x and in the state (γ, μ) suffers a collision while traveling a small distance δs is $\lambda(\gamma, \mu, x)\ \delta s + o(\delta s)$, and the probability of more than one collision is of order $o(\delta s)$. Thus, the collision rate per unit distance traveled λ is seen to be the inverse of the mean free path of the particle. We also assume known the transition probability ϕ for the particle state conditional on a collision; that is, $\phi(A|\gamma_0, \mu_0, x_0)$ is the probability, given a collision at x_0, of a transition from the state (γ_0, μ_0) to some state $(\gamma, \mu) \in A$. Let

$$R_\tau(\mu_0, x_0) = \min\{s | x_0 + \mu_0 s \in \tau\}, \tag{5.1}$$

where $x_0 \in \mathbf{X}_\tau$, and we set $R_\tau(\mu_0, x_0) = \infty$, if $x_0 + \mu_0 s$ does not lie on the surface τ for any finite s. Then it is easy to see that

$$P_0(A \times S_\tau|\gamma_0, \mu_0, x_0) = \exp\left\{- \int_0^{R_\tau(\mu_0, x_0)} \lambda(x_0 + \mu_0 s)\, ds\right\}$$

$$\delta(A|\gamma_0, \mu_0)\ \delta(S_\tau|x_0 + \mu_0 R_\tau(\mu_0, x_0)), \tag{5.2}$$

and it follows from (4.4) that the corresponding no-passage probability

$$\eta = 1 - \kappa_R = \eta_R + \theta_\infty.$$

Thus, if the particle is initially in the state α_0 at x_{Γ_0}, then the no-passage probability $\eta(\Gamma|\alpha_0, x_{\Gamma_0})$ is the sum of two probabilities, $\eta_R(\Gamma|\alpha_0, x_{\Gamma_0})$ and $\theta_\infty(\Gamma|\alpha_0, x_{\Gamma_0})$. Clearly, η_R represents the stopping or escape probability after a finite number of collisions in X_Γ. Hence, θ_∞ must be interpreted as the stopping or escape probability after an infinite number of collisions. Note that the process is stable if and only if $\eta = \eta_R$. It is also clear that $\eta_R = 0$ if and only if the no-passage probability with no collisions $\eta_0 = 0$. Hence, the total no-passage probability $\eta = 0$ if and only if both $\eta_0 = 0$ and $\theta_\infty = 0$.

V. FIRST-PASSAGE DISTRIBUTIONS IN PURE MULTIPLE SCATTERING PROCESSES

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$$R_\Gamma(\mu_0, x_0) = \min\{s \,|\, x_0 + \mu_0 s \in \Gamma\}, \tag{5.1}$$

where $x_0 \in X_\Gamma$, and we set $R_\Gamma(\mu_0, x_0) = \infty$, if $x_0 + \mu_0 s$ does not lie on the surface Γ for any finite s. Then it is easy to see that

$$P_\Gamma(A \times S_\Gamma|\gamma_0, \mu_0, x_0) = \exp\left\{-\int_0^{R_\Gamma(\mu_0, x_0)} \lambda(x_0 + \mu_0 s)\, ds\right\}$$

$$\phi(A|\gamma_0, \mu_0)\, \delta(S_\Gamma|x_0 + \mu_0 R_\Gamma(\mu_0, x_0)), \tag{5.2}$$

where

$$\delta(A\,|\gamma_0,\mu_0) = \begin{cases} 1 \text{ if } (\gamma_0,\mu_0) \in A \\ 0 \text{ otherwise} \end{cases}, \qquad \delta(S_\mathcal{T}|x_0) = \begin{cases} 1 \text{ if } x_0 \in S_\mathcal{T} \\ 0 \text{ otherwise} \end{cases},$$

and we have written, for brevity, x_0 for $x_{\mathcal{T}_0}$, and $\lambda(x_0+\mu_0 s)$ for $\lambda(\gamma_0,\mu_0,x_0+\mu_0 s)$. Similarly, one shows that

$$Q(A\times X;\mathcal{T}|\gamma_0,\mu_0,x_0) = \int_0^{R_\mathcal{T}(\mu_0,x_0)} \phi(A\,|\gamma_0,\mu_0,x_0+\mu_0 s)\;\delta(X\,|x_0+\mu_0 s)$$

$$\cdot \exp\left\{-\int_0^s \lambda(x_0+\mu_0\sigma)\,d\sigma\right\}\lambda(x_0+\mu_0 s)\,ds. \tag{5.3}$$

<u>Lemma 5.1.</u> <u>The distributions P_0 and Q, defined respectively by (5.2) and (5.3), satisfy the consistency relation (4.1).</u>

<u>Proof.</u> Suppose that $\mathcal{T} \geq \mathcal{T}_1 \geq \mathcal{T}_0$, and write x_0 for $x_{\mathcal{T}_0}$, and x_1 for $x_{\mathcal{T}_1}$; then

$$\int_{\mathcal{A}\times\mathcal{T}_1} Q(A\times X;\mathcal{T}|\gamma_1,\mu_1,x_1)\;P_0(d\gamma_1,d\mu_1,dx_1|\gamma_0,\mu_0,x_0)$$

$$= Q\big(A\times X;\mathcal{T}|\gamma_0,\mu_0,x_0+\mu_0 R_{\mathcal{T}_1}(\mu_0,x_0)\big)\;\exp\left\{-\int_0^{R_{\mathcal{T}_1}(\mu_0,x_0)} \lambda(x_0+\mu_0 s)\,ds\right\}$$

$$= \int_0^{R_\mathcal{T}[\mu_0,\,x_0+\mu_0 R_{\mathcal{T}_1}(\mu_0,x_0)]} \phi(A\,|\gamma_0,\mu_0,x_0+\mu_0[R_{\mathcal{T}_1}(\mu_0,x_0)+s])$$

$$\delta\big(X\,x_0+\mu_0[R_{\mathcal{T}_1}(\mu_0,x_0)+s]\big)$$

$$\cdot \exp\left\{-\int_0^s \lambda(x_0+\mu_0\sigma)\,d\sigma - \int_0^{R_{\mathcal{T}_1}(\mu_0,x_0)} \lambda(x_0+\mu_0\sigma)\,d\sigma\right\}$$

$$\lambda\big(x_0+\mu_0[R_{\mathcal{T}_1}(\mu_0,x_0)+s]\big)\;ds. \tag{5.4}$$

From the definition of $R_\mathcal{T}$, it is immediate that

$$R_\mathcal{T}[\mu_0,x_0+\mu_0 R_{\mathcal{T}_1}(\mu_0,x_0)] = R_\mathcal{T}(\mu_0,x_0) - R_{\mathcal{T}_1}(\mu_0,x_0).$$

Substituting this relation in the last line of (5.4) and changing the variables of integration to $\xi = R_{\tau_1}(\mu_0, x_0) + s$, $\zeta = R_{\tau_1}(\mu_0, x_0) + \sigma$, we transform the integral there to:

$$\int_{R_{\tau_1}(\mu_0, x_0)}^{R_\tau(\mu_0, x_0)} \phi(A\,|\gamma_0, \mu_0, x_0 + \mu_0\xi)\ \delta(X\,|x_0 + \mu_0\xi)$$

$$\exp\left\{- \int_{R_{\tau_1}(\mu_0, x_0)}^{\xi} \lambda(x_0 + \mu_0\zeta)\, d\zeta - \int_0^{R_{\tau_1}(\mu_0, x_0)} \lambda(x_0 + \mu_0\zeta)\, d\zeta\right\} \lambda(x_0 + \mu_0\xi)\, d\xi$$

$$= Q(A \times X;\, \tau|\gamma_0, \mu_0, x_0) - Q(A \times X;\, \tau_1|\gamma_0, \mu_0, x_0).$$

This completes the proof of the lemma.

The integral equation (4.2), which the first-passage distribution P must satisfy, becomes in the present case

$$P(A \times S_\tau|\gamma_0, \mu_0, x_0) = P_0(A \times S_\tau|\gamma_0, \mu_0, x_0)$$

$$+ \int_{\mathcal{A}} \int_0^{R_\tau(\mu_0, x_0)} P(A \times S_\tau\,|\gamma, \mu, x_0 + \mu_0 s)\ \phi(d\gamma d\mu|\gamma_0, \mu_0, x_0 + \mu_0 s)$$

$$\cdot \exp\left\{- \int_0^s \lambda(x_0 + \mu_0\sigma)\, d\sigma\right\} \lambda(x_0 + \mu_0 s)\, ds. \tag{5.5}$$

If P satisfies (5.5), then one shows by an elementary calculation, similar to that leading to the so-called "backward" equation (4.26) in I, that P must satisfy the integro-differential equation

$$\mu_0 \cdot \frac{\partial}{\partial x_0} P(\gamma_0, \mu_0, x_0) = \lambda(x_0)\left\{P(\gamma_0, \mu_0, x_0) - \int_{\mathcal{A}} P(\gamma, \mu, x_0)\ \phi(d\gamma d\mu|\gamma_0, \mu_0, x_0)\right\}, \tag{5.6}$$

where we have suppressed the variables $A \times S_\tau$ in the notation for P, and where $\mu_0 \cdot \partial P/\partial x_0$ is the derivative of P in the direction μ_0; i.e.,

$$\mu_0 \cdot \partial P/\partial x_0 = \Sigma_1^3 \mu_0^{(i)} \partial P/\partial x_0^{(i)},$$

$\mu_0^{(i)}$ and $x_0^{(i)}$ being the components of μ_0 and x_0, respectively, in some orthogonal reference frame for R_3.

Substituting this relation in the last line of (5.4) and changing the variable of integration to $\xi = R_{-1}(\mu_0, x_0) + \sigma$, $\zeta = R_{-1}(\mu_0, x_0) + \tau$, we transform the integral there to

$$\int_{-R_{-1}(\mu_0,x_0)}^{0} \psi(A \times S, \mu_0, x_0 + \mu_0\xi)\, \lambda(x_0 + \mu_0\xi)$$

$$\exp\left\{-\int_{-R_{-1}(\mu_0,x_0)}^{\xi} \lambda(x_0 + \mu_0\zeta)\, d\zeta - \int_{0}^{R_{-1}(\mu_0,x_0)} \lambda(x_0 + \mu_0\zeta)\, d\zeta\right\} \lambda(x_0 + \mu_0\xi)\, d\xi$$

$$= Q(A \times S, \tau|\gamma_0, \mu_0, x_0) - Q(A \times X, \tau|\gamma_0, \mu_0, x_0).$$

This completes the proof of the lemma.

The integral equation (4.2), which the first passage distribution P must satisfy, becomes in the present case

$$P(A \times S|\gamma_0, \mu_0, x_0) = P_0(A \times S|\gamma_0, \mu_0, x_0)$$

$$+ \int_0^{R_1(\mu_0,x_0)} \int_{\mathcal{J}} P(A \times S|\gamma, \mu, x_0 + \mu_0 s)\, \theta(d\gamma\, d\mu|\gamma_0, \mu_0, x_0 + \mu_0 s)$$

$$\exp\left\{-\int_0^{s} \lambda(x_0 + \mu_0\sigma)\, d\sigma\right\} \lambda(x_0 + \mu_0 s)\, ds. \qquad (5.5)$$

If P satisfies (5.5), then one shows by an elementary calculation, similar to that leading to the so-called "backward" equation (4.9) in I, that P must satisfy the integro-differential equation

$$\mu_0 \cdot \frac{\partial}{\partial x_0} P(\gamma_0, \mu_0, x_0) = \lambda(x_0)\left\{P(\gamma_0, \mu_0, x_0) - \int_{\mathcal{J}} P(\gamma, \mu, x_0)\, \theta(d\gamma\, d\mu|\gamma_0, \mu_0, x_0)\right\},$$

(5.6)

where we have suppressed the variables $A \times S$ in the notation for P, and where $\mu_0 \cdot \partial P/\partial x_0$ is the derivative of P in the direction μ_0; i.e.,

$$\mu_0 \cdot \partial P/\partial x_0 = \Sigma(\mu_0^{(i)} \partial P/\partial x_0^{(i)}),$$

$\mu_0^{(i)}$ and $x_0^{(i)}$ being the components of μ_0 and x_0, respectively, in some orthogonal reference frame for E_3.

The iteration relations that define the sequences $\{Q_n\}$, $\{P_n\}$, and hence the regular solution P_R, take a simple form in the present case. Thus we have, for the Q_n,

$$Q_{n+1}(A \times X; \tau | \gamma_0, \mu_0, x_0) = \int_{\mathscr{A}} \int_0^{R_\tau(\mu_0, x_0)} Q_n(A \times X; \tau | \gamma, \mu, x_0 + \mu_0 s)$$

$$\cdot\ \phi(d\gamma d\mu | \gamma_0, \mu_0, x_0 + \mu_0 s)$$

$$\exp\left\{- \int_0^s \lambda(x_0 + \mu_0 \sigma)\, d\sigma\right\} \lambda(x_0 + \mu_0 s)\, ds. \tag{5.7}$$

Using the fact that $P_n = P_0 * Q_n$, we find a similar iteration relation for the P_n:

$$P_{n+1}(A \times S_\tau | \gamma_0, \mu_0, x_0) = \int_{\mathscr{A}} \int_0^{R_\tau(\mu_0, x_0)} P_n(A \times S_\tau | \gamma, \mu, x_0 + \mu_0 s)$$

$$\cdot\ \phi(d\gamma d\mu | \gamma_0, \mu_0, x_0 + \mu_0 s)$$

$$\exp\left\{- \int_0^s \lambda(x_0 + \mu_0 \sigma)\, d\sigma\right\} \lambda(x_0 + \mu_0 s)\, ds. \tag{5.8}$$

VI. MULTIPLICATIVE AND CASCADE FIRST-PASSAGE PROCESSES

We now consider first-passage distributions in processes involving the creation and annihilation of particles, as well as their scattering; we shall use the term atomic event as a generic name for all of these. We restrict ourselves here to processes involving only one type of particle (e.g., neutron multiplication, nucleon cascades neglecting meson production). The generalization to processes involving several types of particles (e.g., electron-photon cascades, nucleon cascades with meson production) is immediate and may be effected simply by considering the type of particle as an additional state variable. Suppose that the process starts with k particles (the "ancestors") in specified states and positions inside the surface τ, and let

$$\alpha^{(k)} = (\alpha_{(1)}, \ldots, \alpha_{(k)}),$$

$$x_{\tau_0}^{(k)} = (x_{\tau_0}(1), \ldots, x_{\tau_0}(k))$$

stand, respectively, for these states and positions. As a result of multiplications and annihilations, after an infinite lapse of time, n particles will emerge for the first time through the positions

$$x_\tau^{(n)} = (x_{\tau(1)}, \ldots, x_{\tau(n)})$$

on τ, and in the states

$$\alpha^{(n)} = (\alpha_{(1)}, \ldots, \alpha_{(n)}),$$

where n can be any integer 0, 1, 2, ..., and n = 0 means that no particle effects a first passage through τ. Note that particles may have been created which are either annihilated or stop, or else go to infinity before passing through τ. The first-passage distributions we shall be interested in here are precisely those that yield the probability that n particles will effect a first passage through specified subsets of τ in specified sets of states. Let $\Omega_\tau = \mathscr{A} \times \tau$ be the set of all ordered pairs $\omega_\tau = (\alpha, x_\tau)$. In the present context, an elementary event $\underline{\omega}_\tau = \omega_\tau^{(\cdot)}$ in an ordered set

$$\omega_\tau^{(n)} = (\omega_{\tau(1)}, \ldots, \omega_{\tau(n)});$$

hence, the space of elementary events assigned to τ is

$$\underline{\Omega}_\tau = \bigcup_{n=0}^{\infty} \Omega_\tau^{(n)},$$

where $\Omega_\tau^{(n)}$ is the n-fold Cartesian product of Ω_τ with itself, and $\Omega_\tau^{(0)}$ corresponds to 0 particles. For a fixed initial $\omega_{\tau_0}^{(k)}$, the first-passage distribution $P(\cdot \mid \omega_{\tau_0}^{(k)})$ is then a probability measure on a suitably-defined σ-field of subsets of $\underline{\Omega}_\tau$ normalized as in (2.1). We assume that P is symmetric; i.e., it is invariant under permutations of the coordinates $\omega_{\tau(1)}, \ldots, \omega_{\tau(n)}$ of $\omega_\tau^{(n)}$ for both the initial and the final states, which is equivalent to the assumption that the particles are indistinguishable. We are thus dealing with a stochastic population process [see Moyal[6] for the general theory of such processes]. We shall also assume here that the process is multiplicative [or a branching process; c.f. Harris[4] and Moyal[7]],† in the sense that the k ancestors propagate independently of each other, so that the first-passage distribution $P(\cdot \mid \omega_{\tau_0}^{(1)})$ relative to a single ancestor will suffice to characterize the process. This condition may be expressed more precisely in terms of the probability generating functional (p.g.f. for short) G of P. Let ζ be a bounded measurable function on Ω_τ, and let $\underline{\zeta}$ be the measurable

†The term "cascade process" is used for a multiplicative process whose total energy is nonincreasing.

stand, respectively, for these states and positions. As a result of multiplications and annihilations, after an infinite lapse of time, n particles will emerge for the first time through the positions

$$x_\tau^{(n)} = (x_{\tau(1)}, \ldots, x_{\tau(n)})$$

on τ, and in the states

$$\alpha^{(n)} = (\alpha_{(1)}, \ldots, \alpha_{(n)}),$$

where n can be any integer 0, 1, 2, ..., and $n = 0$ means that no particle effects a first passage through τ. Note that particles may have been created which are either annihilated or stop, or else go to infinity before passing through τ. The first-passage distributions we shall be interested in here are precisely those that yield the probability that n particles will effect a first passage through specified subsets of τ in specified sets of states. Let $\Omega_\tau = \mathcal{A} \times \tau$ be the set of all ordered pairs $\omega_\tau = (\alpha, x_\tau)$. In the present context, an elementary event $\underline{\omega}_\tau = \omega_\tau^{(n)}$ is an ordered set

$$\omega_\tau^{(n)} = (\omega_{\tau(1)}, \ldots, \omega_{\tau(n)});$$

hence, the space of elementary events assigned to τ is

$$\underline{\Omega}_\tau = \bigcup_{n=0}^{\infty} \Omega_\tau^{(n)},$$

where $\Omega_\tau^{(n)}$ is the n-fold Cartesian product of Ω_τ with itself, and $\underline{\Omega}_\tau^{(o)}$ corresponds to 0 particles. For a fixed initial $\omega_{\tau_o}^{(k)}$, the first-passage distribution $P(\cdot\,|\,\omega_{\tau_o}^{(k)})$ is then a probability measure on a suitably-defined σ-field of subsets of $\underline{\Omega}_\tau$ normalized as in (2.1). We assume that P is symmetric, i.e., it is invariant under permutations of the coordinates $\omega_{\tau(1)}, \ldots, \omega_{\tau(n)}$ of $\omega_\tau^{(n)}$ for both the initial and the final states, which is equivalent to the assumption that the particles are indistinguishable. We are thus dealing with a stochastic population process [see Moyal[6] for the general theory of such processes]. We shall also assume here that the process is multiplicative [or a branching process; c.f. Harris[4] and Moyal[7]],[†] in the sense that the k ancestors propagate independently of each other, so that the first-passage distribution $P(\cdot\,|\omega_{\tau_o}^{(1)})$ relative to a single ancestor will suffice to characterize the process. This condition may be expressed more precisely in terms of the probability generating functional (p.g.f. for short) G of P. Let ζ be a bounded measurable function on Ω_τ, and let $\underline{\zeta}$ be the measurable

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[†]The term "cascade process" is used for a multiplicative process whose total energy is nonincreasing.

function on $\underline{\Omega}_\tau$ whose restriction to $\Omega_\tau^{(n)}$ is $\zeta^{(n)}(\omega_\tau^{(n)}) = \zeta(\omega_{\tau(1)}) \dots \zeta(\omega_{\tau(n)})$. Then G is the expectation of $\underline{\zeta}$ relative to P; i.e.,

$$G[\zeta,\tau|\omega_{\tau_c}^{(k)}] = \mathcal{E}(\underline{\zeta}|\omega_{\tau_0}^{(k)}) = \int_{\underline{\Omega}_\tau} \underline{\zeta}(\underline{\omega}_\tau)\, P(d\underline{\omega}_\tau|\omega_{\tau_0}^{(k)})$$

$$= \sum_{n=0}^{\infty} \int_{\Omega_\tau^{(n)}} \zeta(\omega_{\tau(1)}) \dots \zeta(\omega_{\tau(n)})\, P^{(n)}(d\omega_\tau^{(n)}|\omega_{\tau_0}^{(k)}), \tag{6.1}$$

where $P^{(n)}$ is the restriction of P to $\Omega_\tau^{(n)}$. The process is multiplicative if and only if

$$G[\zeta,\tau|\omega_{\tau_0}^{(k)}] = \prod_{i=1}^{k} G[\zeta,\tau|\omega_{\tau_0}(i)]. \tag{6.2}$$

We can now extend to these <u>multiplicative first-passage processes</u> all of the considerations of Sections II, III, IV, and V. Using (6.2), the Chapman-Kolmogorov relation (2.2) becomes, in terms of the p.g.f.,

$$G[\zeta,\tau|\omega_{\tau_0}] = \sum_{n=0}^{\infty} \int_{\Omega^{(n)}} \prod_{i=1}^{n} G[\zeta,\tau|\omega_{\tau_1(i)}]\, P^{(n)}(d\omega_{\tau_1}^{(n)}|\omega_{\tau_0})$$

$$= G\{G[\zeta,\tau|\cdot]|\omega_{\tau_0}\}, \quad \tau \geq 1 \geq \tau_0. \tag{6.3}$$

Similarly, the integral equation (4.2) becomes, in terms of p.g.f.'s,

$$G[\zeta,\tau|\omega_{\tau_0}] = G_0[\zeta,\tau|\omega_{\tau_0}] + \sum_{n=0}^{\infty} \int_{\Omega^{(n)}} \prod_{i=n}^{n} G[\zeta,\tau|\omega_{\tau_1}(i)]\, Q^{(n)}(d\omega_{\tau_1}^{(n)}|\omega_{\tau_0}), \tag{6.4}$$

where G_0 is the p.g.f. of the first-passage distribution P_0 with no atomic events, and Q is the first atomic event position and consequent state distribution. For fixed ω_{τ_0}, $Q(\cdot|\omega_{\tau_0})$ is a probability distribution on a suitably-defined σ-field of subsets of the space

$$\underline{\Omega} = \bigcup_{n=0}^{\infty} (\mathcal{A} \times X)^{(n)}.$$

[$Q^{(0)}$ corresponds to annihilation, $Q^{(1)}$ to scattering, and $Q^{(n)}$ with $n \geq 2$ to creation.] The sequences $\{Q_n\}$, $\{P_n\}$, and the regular solution P_R are defined

as in Section IV; we write $G_n = G_0 * Q_n$ for the p.g.f. of P_n (the first-passage distribution with n atomic events); the p.g.f. of P_R is then

$$G_R = \sum_{n=0}^{\infty} G_n.$$

We generalize Section V in the same way. Let

$$\underline{\mathscr{A}} = \bigcup_{n=0}^{\infty} \mathscr{A}^{(n)};$$

then the transition probability $\phi(\cdot\,|\omega_\tau)$ conditional on an atomic event is a probability measure on a suitably-defined σ-field of subsets of $\underline{\mathscr{A}}$ with $\phi^{(0)}$ corresponding to annihilation, $\phi^{(1)}$ to scattering, and $\phi^{(n)}$, $n \geq 2$, to creation. The integral equation (5.5) takes the form

$$G[\omega_{\tau_0}] = G_0[\omega_{\tau_0}] + \sum_{n=0}^{\infty} \int_{\mathscr{A}^{(n)}} \int_0^{R_\tau(\mu_0, x_0)} \prod_{i=1}^{n} G[\gamma(i), \mu(i), x_0 + \mu_0 s]$$

$$\cdot\ \phi^{(n)}(d\gamma^{(n)} d\mu^{(n)} | \gamma_0, \mu_0, x_0 + \mu_0 s)$$

$$\exp\left\{-\int_0^s \lambda(x_0 + \mu_0\sigma)\, d\sigma\right\} \lambda(x_0 + \mu_0 s)\, ds, \tag{6.5}$$

where, for the sake of brevity, we have suppressed the variables ζ, τ in G and G_0, and the variables γ_0, μ_0 in λ. The "backward" integro-differential equation (5.6) becomes

$$\mu_0 \cdot \frac{\partial}{\partial x_0} G[\gamma_0, \mu_0, x_0] = \lambda(x_0) \left\{ G[\gamma_0, \mu_0, x_0] - \sum_{n=0}^{\infty} \int_{\mathscr{A}^{(n)}} \prod_{i=1}^{n} G[\gamma(i), \mu(i), x_0] \right.$$

$$\left. \cdot\ \phi^{(n)}(d\gamma^{(n)} d\mu^{(n)} | \gamma_0, \mu_0, x_0) \right\}. \tag{6.6}$$

We obtain similar generalizations of the iteration relations (5.7), (5.8), and so on.

In the applications of the foregoing theory, the first-passage distribution with no atomic events P_0 will, as a rule, conserve the total number of particles (i.e., for a single "ancestor,")

$$P_0(\cdot\,|\omega_{\tau_0}) \equiv P_0^{(1)}(\cdot\,|\omega_{\tau_0}).$$

If, however, there is a nonvanishing probability $\eta_0(\tau|\omega_{\tau_0})$ of the particle stopping or escaping to infinity inside τ before the occurrence of the first atomic event, and if we wish the solution G of the integral equation (6.4) to yield the distribution of all particles that eventually effect a first passage through τ, then we must assimilate η_0 to an annihilation probability and include it either in the expression for Q, or in that for P_0; that is, we must either set $Q^{(0)} = q_0 + \eta_0$, where q_0 is the "true" annihilation probability, or we must set $P_0^{(0)} = \eta_0$, and hence

$$G_0[\zeta, \tau|\omega_{\tau_0}] = \eta_0(\tau|\omega_{\tau_0}) + \int_{\Omega_\tau} \zeta(\omega_\tau)\, P_0^{(1)}(d\omega_\tau|\omega_{\tau_0}).$$

In physical applications such as neutron multiplication, where τ represents the boundary of the body where the multiplication occurs, the case in which the probability of an infinity of atomic events $\theta_\infty \neq 0$ will usually mean that the process is "supercritical," since it will usually imply an infinite outgoing flux of particles in the steady state for a constant source inside the body on a constant incoming flux (e.g., see Example 2 in Section VII).

VII. EXAMPLES

Probably the simplest nontrivial examples one can construct to illustrate the foregoing theory are "one-dimensional" ones in which the particles move on a line and the "surfaces" τ are the end-points of intervals. Thus we shall take the set of all surfaces $\mathcal{T}$ to be the set of all pairs of real numbers $\{a, b\}$ with $a < b$, where the "interior" of the surface $\{a, b\}$ is the closed interval $[a, b]$, and possibly, in addition, the set of all real numbers a with "interior" $(-\infty, a]$.

Example 1. The first and simplest example we consider is that of a particle moving with constant absolute velocity, so that the only state variable is the direction of motion μ, which can only take two values: $\mu = 1$ for motion to the right, and $\mu = -1$ for motion to the left. We assume a constant mean free path λ^{-1} and a reversal of the direction of motion at each collision [see Brockwell and Moyal(1) for a more thorough treatment of this example]. We may, without loss of generality, take the "surfaces" to be the set of all pairs $\{-a, a\}$, where $a > 0$. Let $P(\pm a|\mu, x)$ be the probability that the particle initially at x and moving in the direction μ makes a first passage through $\pm a$. Clearly, by symmetry,

$$P(-a|\mu, x) = P(a|-\mu, -x), \tag{7.1}$$

so that one need only determine $P(a|\mu, x)$. The integral equation (5.5) and the corresponding "backward" equation (5.6) become, in this case, respectively, If, however, there is a nonvanishing probability $\pi(\mathbf{r}_0)$ of the particle stopping or escaping to infinity inside Γ before the occurrence of the first atomic event, and if we wish the solution G of the integral equation (6.4) to yield the distribution of all particles that eventually make a first passage through Γ, then we must assimilate π_0 to an annihilation probability and include it either in the expression for Q, or in the collision matrix, i.e., we must either set $Q^{(1)} = q_0 + \pi_0$, where q_0 is the "direct" annihilation probability, or we must set $P_a^{(a)} = \pi_0$, and hence

$$G(\Gamma_1, \omega_1 | \omega_0, r_0) = \pi_0(\Gamma | \omega_0, r_0) + \int_{\Omega_\Gamma} (\omega_1 \Gamma) K_0^{(1)}(d\omega_1 | \omega_0, r_0).$$

In physical applications such as neutron-rocket design, where Γ represents the boundary of the body where the multiplication occurs, the case in which the probability of an infinity of atomic events in Γ is 0 will generally mean that the process is "supercritical," since it will usually imply an infinite outgoing flux of particles in the steady state for a constant source in the body on a constant incoming flux (e.g., see Example 2 in Section VII).

VII. EXAMPLES

Probably the simplest nontrivial examples one can construct to illustrate the foregoing theory are "one-dimensional" ones in which the particles move on a line and the "surfaces" Γ are the endpoints of intervals. Thus we shall take the class of all surfaces Γ to be the set of all pairs of real numbers (a, b) with a < b, where the "interior" of the surface (a, b) is the closed interval [a, b]. In particular, Γ_a [illegible] the set of all real numbers x with "interior" [illegible].

Example 1. The first and simplest example we consider is that of a particle moving with constant absolute velocity, so that the only state variable is the direction of motion μ, which can only take the values μ = +1 for motion to the right, and μ = −1 for motion to the left. We assume a constant mean free path K^{-1} and a reversal of the direction of motion at each collision (see Bartlett and Moyal[1] for a more general treatment of this example). We may, without loss of generality, take the "surfaces" to be the set of all pairs (−a, a), where a > 0. Let F(μ|a, x) be the probability that the particle initially at x and moving in the direction μ makes a first passage through Γ_a. Clearly, by symmetry,

$$F(-\mu | a, x) = F(\mu | a, -x), \qquad (7.1)$$

so that one need only determine F(a|μ, x). The integral equation (5.3) and the corresponding "backward" equation (5.6) become, in this case, respectively,

$$P(a|\mu, x) = e^{-\lambda(a-\mu x)} + \int_0^{a-\mu x} P(a|-\mu, x+\mu s)\, e^{-\lambda s} \lambda ds, \tag{7.2}$$

and

$$\mu \frac{\partial}{\partial x} P(a|\mu, x) = \lambda\{P(a|\mu, x) - P(a|-\mu, x)\}. \tag{7.3}$$

The solution of (7.1) is [c.f. Brockwell and Moyal(1), p. 15]

$$P(a|\mu, x) = [\tfrac{1}{2}(1+\mu) + \lambda(a+x)][1+2\lambda a]^{-1}. \tag{7.4}$$

It follows by (7.2) that

$$P(a|\mu, x) + P(-a|\mu, x) = P(a|\mu, x) + P(a|-\mu, -x) = 1;$$

hence, the process is stable and the solution (7.4) is unique. Equation (7.4) may be obtained either by the iteration relation (5.8), which here takes the form

$$P_{n+1}(a|\mu, x) = \int_0^{a-\mu x} P_n(a|-\mu, x+\mu s)\, e^{-\lambda s} \lambda ds,$$

or by solving (7.3) with the boundary conditions $P(a|1, a) = 1$ and $P(a|-1, -a) = 0$.

<u>Example 2.</u> We construct an example of a multiplicative first-passage process by modifying the previous example as follows: instead of just a reversal at each collision, we assume that there is a probability q_{ij} that the particle splits into $i + j$ particles, with i moving in the same direction, j in the reverse direction, and all with the same, constant, absolute velocity as the "parent" particle, such that

$$\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} q_{ij} = 1.$$

(Note that q_{00} is the probability of annihilation of the "parent," q_{10} the probability of a continuation of, and q_{01} the probability of a reversal of, direction without splitting.) Let $P_{k,n-k}^{(n)}(\mu, x)$ be the probability conditional on a single initial "ancestor" at x and moving in the direction μ that eventually n particles will cross the boundaries a first time, with k moving to the right through a, and $n - k$ moving to the left through $-a$. In this case, the function ζ in the p.g.f. can take only two values: ζ_1 for a passage through a, and ζ_{-1} for a passage through $-a$. The expression (6.1) for the p.g.f. becomes

$$P(a|\mu, x) = e^{-\lambda(a-\mu x)} + \int_0^{a-\mu x} P(a|-\mu, x+\mu s)\, e^{-\lambda s}\lambda\, ds, \qquad (7.2)$$

and

$$\mu \frac{\partial}{\partial x} P(a|\mu, x) = \lambda\{P(a|\mu, x) - P(a|-\mu, x)\}. \qquad (7.3)$$

The solution of (7.1) is [c.f. Brockwell and Moyal[4], p. 15],

$$P(a|\mu, x) = [\tfrac{1}{2}(1+\mu) + \lambda(a+x)][1+2\lambda a]^{-1}. \qquad (7.4)$$

It follows by (7.2) that

$$P(a|\mu, x) + P(-a|\mu, x) = P(a|\mu, x) + P(a|-\mu, -x) = 1;$$

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Example 2. We construct an example of a multiplicative first-passage process by modifying the previous example as follows: instead of just a reversal at each collision, we assume that there is a probability q_{ij} that the particle splits into $i + j$ particles, with i moving in the same direction, j in the reverse direction, and all with the same, constant, absolute velocity as the "parent" particle, such that

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} q_{ij} = 1.$$

[Note that q_{00} is the probability of annihilation of the "parent," q_{10} the probability of a continuation of, and q_{01} the probability of a reversal of, direction without splitting.] Let $P_{k,n-k}^{(a)}(\mu, x)$ be the probability (conditional on a single initial "ancestor" at x and moving in the direction μ) that eventually n particles will cross the boundaries a first time, with k moving to the right through a, and $n - k$ moving to the left through $-a$. In this case, the function ζ in the p.g.f. can take only two values: ζ_1 for a passage through a, and ζ_{-1} for a passage through $-a$. The expression (6.1) for the p.g.f. becomes

$$G[\zeta|\mu, x] = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \zeta_1^k \zeta_{-1}^{n-k} P_{k,n-k}^{(n)}(\mu, x).$$

Let

$$g(\zeta_1, \zeta_{-1}) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \zeta_1^i \zeta_{-1}^j q_{ij};$$

then the integral equation (6.5) and the corresponding "backward" equation (6.6) become, respectively,

$$G[\zeta|\mu, x] = e^{-\lambda(a-\mu x)} + \int_0^{a-\mu x} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} G^i[\zeta|\mu, x+\mu s]\, G^j[\mu|-\mu, x+\mu s]\, e^{-\lambda s}\lambda ds$$

$$= e^{-\lambda(a-\mu x)} + \int_0^{a-\mu x} g(G[\zeta|\mu, x+\mu s], G[\zeta|-\mu, x+\mu s])\, e^{-\lambda s}\lambda ds, \tag{7.5}$$

and

$$\mu \frac{\partial}{\partial x} G[\zeta|\mu, x] = \lambda\{G[\zeta|\mu, x] - g(G[\zeta|\mu, x], G[\zeta|-\mu, x])\}. \tag{7.6}$$

If $q_{-1} = 1$, then equation (7.6) becomes (setting $\lambda = 1$, as we can do without loss in generality, since this merely amounts to taking the mean free path as the unit of length)

$$\mu \frac{\partial}{\partial x} G[\zeta|\mu, x] = G[\zeta|\mu, x]\{1 - G[\zeta|\mu, x]\, G[\zeta|-\mu, x]\}, \tag{7.7}$$

which can be solved explicitly (the solution is due to Mr. P. J. Brockwell). With the boundary conditions $G[\zeta|\mu, \mu a] = \zeta_\mu$, the solution is

$$G[\zeta|\mu, x] = \zeta_\mu \exp\{-[1 - \alpha(\zeta)](a - \mu x)\},$$

where for $0 \le \zeta_\mu \le 1$, $\alpha(\zeta)$ is the minimal nonnegative solution of the functional equation

$$\alpha(\zeta) \exp\{2a[1 - \alpha(\zeta)]\} = \zeta_1 \zeta_{-1}.$$

The probability of an infinity of atomic events is then

$$\theta_\infty(\mu, x) = 1 - \exp\{-(1-\gamma)(a - \mu x)\},$$

$$G(\zeta|\mu,x) = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\zeta_1^{k}\zeta_{-1}^{n-k}P^{(n)}_{k,n-k}(\mu,x).$$

Let

$$g(\zeta_1,\zeta_{-1}) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} q_{ij}\,\zeta_1^{i}\,\zeta_{-1}^{j};$$

then the integral equation (6.5) and the corresponding "backward" equation (6.6) become, respectively,

$$G[\zeta|\mu,x] = e^{-\lambda(a+\mu x)} + \int_0^{a-\mu x}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} q_{ij}\,G^{i}[\zeta|\mu,x+\mu s]\,G^{j}[\zeta|-\mu,x+\mu s]\,e^{-\lambda s}\lambda ds$$

$$= e^{-\lambda(a-\mu x)} + \int_0^{a-\mu x} g(G[\zeta|\mu,x+\mu s],\,G[\zeta|-\mu,x+\mu s])\,e^{-\lambda s}\lambda ds,$$

(7.5)

and

$$\mu\frac{\partial}{\partial x}G[\zeta|\mu,x] = \lambda\{G[\zeta|\mu,x] - g(G[\zeta|\mu,x],\,G[\zeta|-\mu,x])\}. \quad (7.6)$$

If $q_{11} = 1$, then equation (7.6) becomes (setting $\lambda = 1$, as we can do without loss in generality, since this choice amounts to taking the mean free path as the unit of length)

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$$G[\zeta|\mu,x] = \zeta_\mu\exp\{-[1-\alpha(\zeta)](a-\mu x)\},$$

where for $0 \le \zeta_1\zeta_{-1} \le 1$, $\alpha(\zeta)$ is the minimal non-negative solution of the functional equation

$$\alpha(\zeta)\exp\{2a[1-\alpha(\zeta)]\} = \zeta_1\zeta_{-1}.$$

The probability of an infinity of atomic events is then

$$\theta_\mu(\mu,x) = 1 - \exp\{-(1-\gamma)(a-\mu x)\}.$$

where $\gamma = \alpha(1)$ is the smallest nonnegative solution of

$$\gamma \exp[2a(1-\gamma)] = 1;$$

hence, $\gamma = 1$, and the process is stable (i.e., $\theta_\infty \equiv 0$) if and only if $2a \le 1$; otherwise $\gamma < 1$, and the process is unstable. We may therefore interpret $2a = 1$ as the "criticality" condition for this process and say that it is "subcritical" when $2a < 1$, "critical" when $2a = 1$, and "supercritical" when $2a > 1$. The total mean number of particles which effect a first passage through either a or -a is (setting $\zeta_1 = \zeta_{-1} = z$)

$$m(\mu, x) = \left\{\frac{\partial}{\partial z} G[z|\mu, x]\right\}_{z=1} = \left\{1 + \frac{2(a-\mu x)}{1-2a\gamma}\right\} \exp\{-(1-\gamma)(a-\mu x)\}. \tag{7.8}$$

We see that this mean is infinite when the process is critical, or more precisely, that $m \to +\infty$ if $\mu x \neq a$ and $2a \to 1$ from the left. If we set $\gamma = 1$ in (7.8), then

$$m(\mu, x) = \frac{1-2\mu x}{1-2a}, \tag{7.9}$$

which is the solution of the equation for the mean

$$\mu \frac{\partial}{\partial x} m(\mu, x) = -m(\mu, x) - m(-\mu, x) \tag{7.10}$$

with boundary conditions $m(\mu, \mu a) = 1$, for all values of a. The reason for the discrepancy between (7.8) and (7.9) is simply that equation (7.10) is obtained by setting $\zeta_1 = \zeta_{-1} = z$ in (7.7), differentiating both sides with respect to z, and then setting $z = 1$ and $G[1|\mu, x] \equiv 1$. It therefore ceases to be valid in the supercritical case where

$$G[1|\mu, x] = \kappa_R(\mu, x) = 1 - \theta_\infty(\mu, x) \not\equiv 1,$$

and this is reflected by the fact that m in (7.9) can take negative values when $2a > 1$.

<u>Example 3.</u> We will now exhibit an example of an unstable, one-dimensional, purely scattering process. We assume (1) that the mean free path is v/α, where α is a constant and v is the absolute velocity of the particle; (2) that at each collision there is a constant probability p that the direction of motion μ is reversed, and $1 - p$ that it continues the same; and (3) that the absolute velocity u after a collision is uniformly distributed between 0 and the velocity v before the collision, independently of whether the direction of motion is reversed or continued. Let $P(u, \pm a|v, \mu, x)\,du$ be the probability that the particle, initially at x with velocity v and direction of motion μ,

where $\gamma = \gamma(1)$ is the smallest nonnegative solution of

$$\gamma \exp[2a(1-\gamma)] = 1;$$

hence, $\gamma = 1$, and the process is stable (i.e., $\theta_\infty = 0$) if and only if $2a \le 1$; otherwise $\gamma < 1$, and the process is unstable. We may therefore interpret $2a = 1$ as the "criticality" condition for this process and say that it is "subcritical" when $2a < 1$, "critical" when $2a = 1$, and "supercritical" when $2a > 1$. The total mean number of particles which effect a first passage through either a or -a is (setting $\zeta_1 = \zeta_{-1} = z$),

$$m(\mu, x) = \left\{\frac{\partial}{\partial z} G[z|\mu, x]\right\}_{z=1} = \left\{1 + \frac{2(a-\mu x)}{1-2a\gamma}\right\}\exp\{(1-\gamma)(a-\mu x)\}. \qquad (7.8)$$

We see that this mean is infinite when the process is critical, or more precisely, that $m \to +\infty$ if $\mu x \neq a$ and $2a \to 1$ from the left. If we set $\gamma = 1$ in (7.8), then

$$m(\mu, x) = \frac{1-2\mu x}{1-2a} \qquad (7.9)$$

which is the solution of the equation for the mean

$$\mu\frac{\partial}{\partial x} m(\mu, x) = -m(\mu, x) - m(-\mu, x) \qquad (7.10)$$

with boundary conditions $m(\mu, \mu a) = 1$, for all values of a. The reason for the discrepancy between (7.8) and (7.9) is simply that equation (7.10) is obtained by setting $\zeta_1 = \zeta_{-1} = z$ in (7.7), differentiating both sides with respect to z, and then setting z = 1 and $G[1|\mu, x] \equiv 1$. It therefore ceases to be valid in the supercritical case where

$$G[1|\mu, x] = \kappa g(\mu, x) = 1 - \theta_\infty(\mu x) \neq 1,$$

and this is reflected by the fact that m in (7.9) can take negative values when $2a > 1$.

Example 3. We will now exhibit an example of an unstable, one-dimensional, purely scattering process. We assume (1) that the mean free path is $v\alpha$, where α is a constant and v is the absolute velocity of the particle; (2) that at each collision there is a constant probability, p that the direction of motion μ is reversed, and 1 - p that it continues the same; and (3) that the absolute velocity u after a collision is uniformly distributed between 0 and the velocity v before the collision, independently of whether the direction of motion is reversed or continued. Let $P(u, x | v, \mu)\, du$ be the probability that the particle, initially at x with velocity v and direction of motion μ, makes a first-passage through $\pm a$ with velocity between u and u + du; clearly, P = 0 when u > v and P satisfies (7.1). The integral and "backward" equations for this process are, respectively,

$$P(u, a|\sigma, \mu, x) = \exp\left[-\frac{\alpha}{v}(a-x)\right] + \int_0^{a-\mu x} \exp\left[-\frac{\alpha s}{v}\right]\frac{\alpha}{v}\,ds$$

$$\int_u^v \{pP(u, a|w, -\mu, x+\mu s) + (1-p)\,P(u, a|w, u, x+\mu s)\}\frac{dw}{v},$$

and

$$\mu\frac{\partial}{\partial x}P(u, a|v, \mu, x) = \frac{\alpha}{v}P(u, a|v, \mu, x)$$

$$-\int_u^v \{pP(u, a|w, -\mu, x) + (1-p)\,P(u, a|w, \mu, x)\}\frac{dw}{v}.$$

Let $\theta_\infty^{(p)}$ be the probability of an infinity of collisions for a given p. In the degenerate case p = 0, $\theta_\infty^{(0)}$ can be obtained explicitly (see I, p. 259):

$$\theta_\infty^{(0)}(v, \mu, x) = 1 - \left[1+\frac{\alpha}{v}(a-\mu x)\right]\exp\left[-\frac{\alpha}{v}(a-\mu x)\right] \not\equiv 0.$$

It can be shown that

$$\theta_\infty^{(p)}(v, 1, x) \geq \theta_\infty^{(0)}(v, 1x)$$

for $x \geq 0$ and $0 \leq p \leq 1$, which proves that this process is unstable for all p.

The collision rate per unit time of this process is the constant α, so that the probability of n collisions in a finite time interval t is

$$p_n^{(t)} = (\alpha t)^n\, e^{-\alpha t}/n!;$$

hence,

$$\Sigma_0^\infty p_n(t) = 1,$$

which means that the probability of an infinite number of collisions is zero in any finite time interval. However, each collision slows the particle down and thereby decreases its mean free path, thus creating the possibility that it will not reach either boundary in any finite time. The probability that the makes a first-passage through (a) with velocity between u and u + du; clearly, P = 0 when u > v and P satisfies (7.1). The integral and "backward" equations for this process are, respectively,

$$F(u, a|v, \mu, x) = \exp\left[-\frac{\alpha}{v}(a-x)\right] + \int_0^{a-\mu x} \exp\left[-\frac{\alpha s}{v}\right]\frac{\alpha}{v}\, ds$$

$$\int_u^v \{pF(u, a|w, -\mu, x+\mu s) + (1-p)\, F(u, a|w, \mu, x+\mu s)\}\,\frac{dw}{v}$$

and

$$\mu\frac{\partial}{\partial x}F(u, a|v, \mu, x) = \frac{\alpha}{v}F(u, a|v, \mu, x)$$

$$-\int_u^v \{pF(u, a|w, -\mu, x) + (1-p)\, F(u, a|w, \mu, x)\}\,\frac{dw}{v}$$

Let $\theta_\infty^{(p)}$ be the probability of an infinity of collisions for a given p. In the degenerate case p = 0, $\theta_\infty^{(0)}$ can be obtained explicitly (see 1, p. 249):

$$\theta_\infty^{(0)}(v, \mu, x) \geq 1 - \left[1+\frac{\alpha}{v}(a-\mu x)\right]\exp\left[-\frac{\alpha}{v}(a-\mu x)\right] \neq 0.$$

It can be shown that

$$\theta_\infty^{(p)}(v, 1, x) \geq \theta_\infty^{(0)}(v, |x|)$$

for x ≥ 0 and 0 ≤ p ≤ 1, which proves that this process is unstable* for all p.

The collision rate per unit time of this process is the constant α, so that the probability of n collisions in a finite time interval t is

$$p_n^{(t)} = (\alpha t)^n\, e^{-\alpha t}/n!$$

hence,

$$\sum_0^\infty p_n(t) = 1,$$

which means that the probability of an infinite number of collisions is zero in any finite time interval. However, each collision slows the particle down and thereby decreases its mean free path, thus creating the possibility that it will not reach either boundary in any finite time. The probability that the particle is thus stopped is precisely θ_∞, and since

$$\lim_{t\to\infty} \Sigma_0^n p_j(t) = 0 \text{ for all } n,$$

we see that θ_∞ is also the probability that the particle will suffer an infinite number of collisions.

Example 4. The Milne Problem. We shall now consider briefly the Milne problem from the point of view of the present paper; the treatment is very similar to that of Sobolev(9), Ch. 6. We are concerned with a particle moving with constant velocity and constant mean free path (which we can take equal to unity) in R_3, and suffering isotropic scattering at each collision. We take $\mathscr{T}$ to see the set of all ordered pairs of planes normal to the x-axis, and we can, without loss of generality, assume that they cut this axis at $\pm a$, where $a > 0$. Let θ be the angle between the direction of motion and the x-axis; let $\mu = |\cos\theta|$ and $\sigma = \operatorname{sgn}(\cos\theta)$. We see by symmetry considerations that the first-passage probability density for the planes $\pm a$ depends only on the initial values σ, μ, and k, on the final values σ_a and μ_a, and on

$$r_a = \sqrt{(y_a - y)^2 + (z_a - z)^2},$$

where P refers to a first passage through the planes $\pm a$ according as $\sigma_a = \pm 1$, and y_a, z_a are the coordinates of the point of passage. Let

$$B(\sigma_a, \mu_a, r_a|x) = \tfrac{1}{2}\sum_{\sigma=-1}^{1}\int_0^1 P(\sigma_a, \mu_a, r_a|\sigma, \mu, x)\, d\mu,$$

and let B_n be similarly related to P_n. The integral equation (5.5) becomes

$$P(\sigma_a, \mu_a, r_a|\sigma, \mu, x) = P_0(\sigma_a, \mu_a, r_a|\sigma, \mu, x) + \int_0^{(a-\sigma x)/\mu} B(\sigma_a, \mu_a, r_a|x+\sigma s)\, e^{-s} ds, \tag{7.11}$$

where

$$P_0(\sigma_a, \mu_a, r_a|\sigma, \mu, x) = \delta(\mu_a - \mu)\, \delta\left[r_a - (a - \sigma x)\sqrt{1-\mu^2}\right] e^{-(a-\sigma x)/\mu}.$$

It follows that B satisfies the integral equation

$$B(\sigma_a, \mu_a, r_a|x) = B_0(\sigma_a, \mu_a, r_a|x) + \int_{-a}^{a} B(\sigma_a, \mu_a, r_a|\xi)\, \mathrm{Ei}(\xi - x)\, d\xi, \tag{7.12}$$

particle is thus stopped is precisely θ_∞, and since

$$\lim_{t\to\infty} \Sigma^n p(t) = 0 \text{ for all } n,$$

we see that θ_∞ is also the probability that the particle will suffer an infinite number of collisions.

Example 4. The Milne Problem. We shall now consider briefly the Milne problem from the point of view of the present paper; the treatment is very similar to that of Sobolev[9], Ch. 6. We are concerned with a particle moving with constant velocity and constant mean free path (which we can take equal to unity) in E_3, and suffering isotropic scattering at each collision. We take $\mathcal{F}$ to be the set of all ordered pairs of planes normal to the x-axis and we can, without loss of generality, assume that they cut this axis at $\pm a$, where $a > 0$. Let θ be the angle between the direction of motion and the x-axis, let $\mu = |\cos\theta|$ and $\sigma = \operatorname{sgn}\cos\theta$. We see by symmetry considerations that the first-passage probability density for the planes $\pm a$ depends only on the initial values σ, μ, and x, on the final values σ_a and μ_a, and on

$$r_a = \sqrt{(y_a - y)^2 + (z_a - z)^2},$$

where P refers to a first passage through the planes $\pm a$ according as $\sigma_a = \pm 1$, and y_a, z_a are the coordinates of the point of passage. Let

$$Q(\sigma_a,\mu_a,r_a|x) = \frac{1}{2}\sum_{\sigma=\pm 1}\int_0^1 P(\sigma_a,\mu_a,r_a|\sigma,\mu,x)\,d\mu,$$

and let R_0 be similarly related to P_0. The integral equation (5.5) becomes

$$P(\sigma_a,\mu_a,r_a|\sigma,\mu,x) = P_0(\sigma_a,\mu_a,r_a|\sigma,\mu,x) + \int_0^{(a-\sigma x)/\mu} Q(\sigma_a,\mu_a,r_a|x+\sigma s)\,e^{-s}\,ds, \quad (7.11)$$

where

$$P_0(\sigma_a,\mu_a,r_a|\sigma,\mu,x) = \delta(\sigma_a-\sigma)\,\delta(\mu_a-\mu)\,\delta\left[r_a - (a-\sigma x)\sqrt{1-\mu^2}\right]e^{-(a-\sigma x)/\mu}.$$

It follows that Q satisfies the integral equation

$$Q(\sigma_a,\mu_a,r_a|x) = R_0(\sigma_a,\mu_a,r_a|x) + \int_{-a}^{a} Q(\sigma_a,\mu_a,r_a|\xi)\,E(|\xi-x|)\,d\xi. \quad (7.12)$$

where

$$\mathrm{Ei}(x) = \int_0^1 e^{-|x|/\mu} \frac{d\mu}{\mu}.$$

It then follows, from the iteration relation (5.8), that

$$B_{n+1}(\sigma_a, \mu_a, r_a | x) = \int_{-a}^{a} B_n(\sigma_a, \mu_a, r_a | \xi) \, \mathrm{Ei}(\xi - x) \, d\xi ,$$

and that

$$B = \sum_{n=0}^{\infty} B_n$$

is the solution of (7.12) (it is easy to see that the series converges). The first passage distribution P is then obtained from B by (7.11).

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